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Gram-type pfaffian solution to the coupled discrete KP equation

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Abstract
In this paper, we first prove the Grammian determinant solution to the discrete KP (dKP) equation by the algebraic identity of pfaffian instead of Laplace expansion for determinants. Then we present the Gram-type pfaffian solution to the pfaffianized dKP system. As an example, the N-soliton solution for the system is obtained.

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1. Introduction

In the early 1990s, a procedure for generalizing equations from the KP hierarchy was developed to produce coupled systems of equations which is now called pfaffianization [1, 2]. These pfaffianized equations appear as coupled systems of the original equations and have soliton solutions expressed by pfaffians. Such a procedure has been successfully applied to the Davey–Stewartson equations [3], the self-dual Yang–Mills equation [4] and the dKP equation [5], etc.

In the studies of KP hierarchy, the Wronskian determinant solutions first appeared in [6–9] and later the Grammian determinant solutions for the KP equation were investigated in [2, 10]. In [1], the Gram-type pfaffian solutions for the coupled KP equation were also considered. Following the KP equation and the coupled KP equation, in [11], we have obtained the Grammian determinant solutions and the Gram-type pfaffian solutions to three pfaffianized systems derived from the two-dimensional Toda equation, the semi-discrete Toda equation and the differential-difference KP equation successfully.

As for a fully discrete equation, the Grammian determinant solution to the dKP equation has been studied in [12]. Therefore we can consider the discrete Gram-type pfaffian solution for the pfaffianized dKP equation in the same way.

The paper is organized as follows. In section 2, we give another proof of the Grammian determinant solution to the dKP equation by the algebraic identity of a pfaffian. In section 3, we present the Gram-type pfaffian solution to the pfaffianized dKP system. Section 4 is a conclusion.

2. Elementary properties of a pfaffian

For the mathematics of pfaffians, the readers may refer to some textbooks [13]. In this section, we summarize some properties of pfaffians playing important roles in the solition theory. In the following, we write the (i, j) element of a pfaffian simply as (i, j).

Pfaffians are antisymmetric functions with respect to characters,

$$(i, j) = -(j, i), \qquad \text{for any} \quad i \quad \text{and} \quad j.$$

A pfaffian of degree $2n$ is inductively defined by the following expansion rule:

$$(1, 2, \ldots, 2n) = \sum_{j=2}^{2n} (-1)^j (1, j)(2, 3, \ldots, \hat{j}, \ldots, 2n),$$

where $\hat{j}$ denotes the missing of the letter j. For example, if $n = 2$, we have

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).$$

From the definition of a pfaffian, we can see that pfaffians are closely related to determinants. A determinant of degree n

$$B = \det|b_{j,k}|_{1\leqslant j,k\leqslant n}$$

can be expressed by means of a pfaffian of degree $2n$ as

$$\det|b_{j,k}|_{1\leqslant j,k\leqslant n} = (1, 2, \ldots, n, n^*, \ldots, 2^*, 1^*),$$

whose entries (j, k), (j^*, k^*) and (j, k^*) are defined by

$$(j, k) = 0, \qquad (j^*, k^*) = 0, \qquad (j, k^*) = b_{j,k}.$$

The most important property of a pfaffian is given in the following golden theorem for the expansion of a pfaffian.

Theorem. *If an identity on pfaffians holds, the equation which is obtained by appending* $(1, 2, \ldots, 2n)$ *homogenously to all pfaffians in the identity also holds.*

Example. We have

$$(a_1, a_2, a_3, a_4) = (a_1, a_2)(a_3, a_4) - (a_1, a_3)(a_2, a_4) + (a_1, a_4)(a_2, a_3).$$

Applying the above theorem, we have

$$\begin{aligned} &\mathrm{pf}(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2n)\,\mathrm{pf}(1, 2, \ldots, 2n) \\ &\quad = \mathrm{pf}(a_1, a_2, 1, 2, \ldots, 2n)\,\mathrm{pf}(a_3, a_4, 1, 2, \ldots, 2n) \\ &\qquad - \mathrm{pf}(a_1, a_3, 1, 2, \ldots, 2n)\,\mathrm{pf}(a_2, a_4, 1, 2, \ldots, 2n) \\ &\qquad + \mathrm{pf}(a_1, a_4, 1, 2, \ldots, 2n)\,\mathrm{pf}(a_2, a_3, 1, 2, \ldots, 2n). \end{aligned} \tag{1}$$

As a special case, we get the Jacobi identity for the determinant

$$\begin{aligned} &(a_1, b_1, a_2, b_2, 1, \ldots, n, n^*, \ldots, 1^*)(1, \ldots, n, n^*, \ldots, 1^*) \\ &\quad = (a_1, b_1, 1, \ldots, n, n^*, \ldots, 1^*)(a_2, b_2, 1, \ldots, n, n^*, \ldots, 1^*) \\ &\qquad + (a_1, b_2, 1, \ldots, n, n^*, \ldots, 1^*)(b_1, a_2, 1, \ldots, n, n^*, \ldots, 1^*), \end{aligned} \tag{2}$$

where

$$(a_1, a_2) = (b_1, b_2) = 0, \qquad (a_i, j) = 0, \qquad (b_i, j^*) = 0, \qquad (i, j) = (i^*, j^*) = 0.$$

3. Discrete grammian determinant solution to the dKP equation

The discrete KP equation in bilinear form is

$$\begin{aligned} &a_1(a_2 - a_3)\tau(k_1 + a_1, k_2, k_3)\tau(k_1, k_2 + a_2, k_3 + a_3) \\ &\quad + a_2(a_3 - a_1)\tau(k_1, k_2 + a_2, k_3)\tau(k_1 + a_1, k_2, k_3 + a_3) \\ &\quad + a_3(a_1 - a_2)\tau(k_1, k_2, k_3 + a_3)\tau(k_1 + a_1, k_2 + a_2, k_3) = 0. \end{aligned} \tag{3}$$

In [12], the discrete Gram-type determinant solution was presented and has been proved by the Laplace expansion for determinants. In this section, we will prove the solution by the Jacobi identity.

The discrete Gram-type determinant solution for the discrete KP equation is

$$\tau(k_1, k_2, k_3) = \det(m_{ij}(k_1, k_2, k_3))_{1\leqslant i,j\leqslant N}, \tag{4}$$

where the matrix element m_{ij} is an arbitrary function of k_1, k_2, k_3 satisfying

$$\Delta_{+k_\nu} m_{ij}(k_1, k_2, k_3) = \phi_i(k_\nu + a_\nu; 0)\bar{\phi}_j(0), \qquad \nu = 1, 2, 3. \tag{5}$$

In the above, ϕ_i and $\bar{\phi}_i$ are arbitrary functions of k_1, k_2, k_3 and s, satisfying the dispersion relations

$$\Delta_{-k_\nu}\phi_i(k_1, k_2, k_3; s) = \phi_i(k_1, k_2, k_3; s + 1), \tag{6}$$

$$\Delta_{+k_\nu}\bar{\phi}_i(k_1, k_2, k_3; s) = \bar{\phi}_i(k_1, k_2, k_3; s + 1), \tag{7}$$

where Δ_{-k_ν} and Δ_{+k_ν} are the backward and forward difference operators which are defined by

$$\Delta_{-k_\nu} F(k_\nu) = \frac{F(k_\nu) - F(k_\nu - a_\nu)}{a_\nu}, \tag{8}$$

$$\Delta_{+k_\nu} F(k_\nu) = \frac{F(k_\nu + a_\nu) - F(k_\nu)}{a_\nu}. \tag{9}$$

Hereafter, the unshifted independent variables are omitted and only the shifted variables are denoted. In [12], the N-soliton solution for equation (3) was also presented.

As we know, any determinant can be expressed by a pfaffian. Thus we have

$$\tau = (1, \ldots, N, N^*, \ldots, 1^*), \qquad (i, j) = (i^*, j^*) = 0, \tag{10}$$

$$\Delta_{+k_\nu}(i, j^*) = \frac{(i, j^*)_{k_\nu + a_\nu} - (i, j^*)}{a_\nu} = \phi_i(k_\nu + a_\nu; 0)\bar{\phi}_j(0), \tag{11}$$

where ϕ_i and $\bar{\phi}_i$ satisfy the dispersion relations (equations (6) and (7)). In order to prove that τ satisfies equation (3), let us introduce pfaffians defined by

$$(d_\nu^*, i) = -\phi_i(k_\nu + a_\nu; 0), \qquad (\bar{d}_n, j^*) = -\bar{\phi}_j(n), \qquad (\bar{d}_n, d_\nu^*) = (-1)^n \frac{1}{a_\nu^{n+1}}, \tag{12}$$

$$(\bar{d}_m, \bar{d}_n) = (d_\mu^*, d_\nu^*) = 0, \qquad (\bar{d}_n, j) = 0, \qquad (d_\nu^*, j^*) = 0. \tag{13}$$

Denoting $\tau = (1, \ldots, N, N^*, \ldots, 1^*) = (\bullet)$, by employing the above pfaffians and equations (6) and (7), we have

$$\frac{1}{a_\nu}\tau(k_\nu + a_\nu) = (\bar{d}_0, d_\nu^*, \bullet), \tag{14}$$

$$\frac{a_\nu - a_\mu}{(a_\nu a_\mu)^2}\tau(k_\nu + a_\nu, k_\mu + a_\mu) = (\bar{d}_1, \bar{d}_0, d_\mu^*, d_\nu^*, \bullet), \qquad \nu, \mu = 1, 2, 3, \qquad \nu < \mu. \tag{15}$$

Substituting equations (14) and (15) into equation (3) leads to the following identity:

$$(\bar{d}_0, d_1^*, \bullet)(\bar{d}_1, \bar{d}_0, d_3^*, d_2^*, \bullet) - (\bar{d}_0, d_2^*, \bullet)(\bar{d}_1, \bar{d}_0, d_3^*, d_1^*, \bullet) + (\bar{d}_0, d_3^*, \bullet)(\bar{d}_1, \bar{d}_0, d_2^*, d_1^*, \bullet) = 0. \tag{16}$$

In fact, according to the pfaffian identity, the left-hand side of equation (16) is equal to $(\bar{d}_0, \bar{d}_1, \bar{d}_0, d_3^*, d_2^*, d_1^*, \ldots)(\cdots)$ which is obviously 0.

4. Gram-type pfaffian solution to the coupled dKP equation

The idea of the pfaffianization is just removing the condition of the specialization $(i, j) = (i^*, j^*) = 0$ in equation (10) and considering the generic form of pfaffian elements. Then we can get the pfaffianized bilinear equations which are satisfied by those pfaffians. In [5], a pfaffianized dKP system is obtained:

$$\begin{aligned} &a_1(a_2 - a_3)\tau(k_1 + a_1, k_2, k_3)\tau(k_1, k_2 + a_2, k_3 + a_3) \\ &\quad + a_2(a_3 - a_1)\tau(k_1, k_2 + a_2, k_3)\tau(k_1 + a_1, k_2, k_3 + a_3) \\ &\quad + a_3(a_1 - a_2)\tau(k_1, k_2, k_3 + a_3)\tau(k_1 + a_1, k_2 + a_2, k_3) \\ &= a_1a_2a_3(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)\sigma(k_1 + a_1, k_2 + a_2, k_3 + a_3)\bar{\sigma}(k_1, k_2, k_3), \end{aligned} \tag{17}$$

$$\begin{aligned} &a_2a_3(a_2 - a_3)\tau(k_1 + a_1, k_2, k_3)\sigma(k_1, k_2 + a_2, k_3 + a_3) \\ &\quad + a_1a_3(a_3 - a_1)\tau(k_1, k_2 + a_2, k_3)\sigma(k_1 + a_1, k_2, k_3 + a_3) \\ &\quad + a_1a_2(a_1 - a_2)\tau(k_1, k_2, k_3 + a_3)\sigma(k_1 + a_1, k_2 + a_2, k_3) \\ &\quad + (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)\sigma(k_1 + a_1, k_2 + a_2, k_3 + a_3)\tau(k_1, k_2, k_3) = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &a_2a_3(a_2 - a_3)\bar{\sigma}(k_1 + a_1, k_2, k_3)\tau(k_1, k_2 + a_2, k_3 + a_3) \\ &\quad + a_1a_3(a_3 - a_1)\bar{\sigma}(k_1, k_2 + a_2, k_3)\tau(k_1 + a_1, k_2, k_3 + a_3) \\ &\quad + a_1a_2(a_1 - a_2)\bar{\sigma}(k_1, k_2, k_3 + a_3)\tau(k_1 + a_1, k_2 + a_2, k_3) \\ &\quad + (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)\tau(k_1 + a_1, k_2 + a_2, k_3 + a_3)\bar{\sigma}(k_1, k_2, k_3) = 0. \end{aligned} \tag{19}$$

Following the coupled KP equation, we can get the following Gram-type pfaffian solution for the above pfaffianized system:

$$\tau = (1, \ldots, 2N), \qquad \sigma = (d_0, d_1, 1, \ldots, 2N), \qquad \bar{\sigma} = (\bar{d}_0, \bar{d}_1, 1, \ldots, 2N), \tag{20}$$

where the (i, j) element is an arbitrary function of k_1, k_2, k_3 satisfying

$$\Delta_{+k_\nu}(i, j) = \frac{(i, j)_{k_\nu + a_\nu} - (i, j)}{a_\nu} = \phi_i(k_\nu + a_\nu; 0)\bar{\phi}_j(0) - \phi_j(k_\nu + a_\nu; 0)\bar{\phi}_i(0), \tag{21}$$

where ϕ_i and $\bar{\phi}_i$ are arbitrary functions of k_1, k_2, k_3 and s, satisfying the dispersion relations equations (6) and (7), and other pfaffian elements are given by

$$(d_n, i) = -\phi_i(n), \qquad (d_0, d_1) = 0, \qquad (d_0, c_\nu) = 0, \qquad (\bar{d}_n, i) = -\bar{\phi}_i(n), \tag{22}$$

$$(\bar{d}_n, \bar{d}_m) = (c_\nu, c_\mu) = 0, \qquad (c_\nu, i) = -\phi_i(k_\nu + a_\nu; 0), \qquad (\bar{d}_n, c_\nu) = (-1)^n \frac{1}{a_\nu^{n+1}}. \tag{23}$$

By employing the above pfaffians and the dispersion relations (21), denoting $(1, 2, \ldots, 2N) = (\bullet)$, we have

$$\frac{1}{a_\nu}\tau(k_\nu + a_\nu) = (\bar{d}_0, c_\nu, \bullet), \qquad \nu = 1, 2, 3, \tag{24}$$

$$\frac{a_\nu - a_\mu}{(a_\nu a_\mu)^2}\tau(k_\nu + a_\nu, k_\mu + a_\mu) = (\bar{d}_1, \bar{d}_0, c_\mu, c_\nu, \bullet), \qquad \nu, \mu = 1, 2, 3, \quad \nu < \mu, \tag{25}$$

$$\frac{(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)}{(a_1a_2a_3)^3}\tau(k_1 + a_1, k_2 + a_2, k_3 + a_3) = (\bar{d}_2, \bar{d}_1, \bar{d}_0, c_3, c_2, c_1, \bullet), \tag{26}$$

$$a_\nu\sigma(k_\nu + a_\nu) = (d_0, c_\nu, \bullet), \qquad \nu = 1, 2, 3, \tag{27}$$

$$(a_\nu - a_\mu)\sigma(k_\nu + a_\nu, k_\mu + a_\mu) = (c_\mu, c_\nu, \bullet), \qquad \nu, \mu = 1, 2, 3, \quad \nu < \mu, \tag{28}$$

$$\frac{(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)}{a_1a_2a_3}\sigma(k_1 + a_1, k_2 + a_2, k_3 + a_3) = (\bar{d}_0, c_3, c_2, c_1, \bullet), \tag{29}$$

$$\frac{1}{a_\nu^3}\bar{\sigma}(k_\nu + a_\nu) = (\bar{d}_0, \bar{d}_1, \bar{d}_2, c_\nu, \bullet), \qquad \nu = 1, 2, 3. \tag{30}$$

Hereafter, we denote $\tau = (1, \ldots, 2N) = (\bullet)$. Substituting the above pfaffians into equations (17)–(19), we get the following pfaffian identities:

$$\begin{aligned} 0 &= (\bar{d}_0, \bar{d}_1, \bar{d}_0, c_3, c_2, c_1, \bullet)(\bullet) \\ &= (\bar{d}_0, \bar{d}_1, \bullet)(\bar{d}_0, c_3, c_2, c_1, \bullet) + (\bar{d}_0, c_3, \bullet)(\bar{d}_1, \bar{d}_0, c_2, c_1, \bullet) \\ &\quad - (\bar{d}_0, c_2, \bullet)(\bar{d}_1, \bar{d}_0, c_3, c_1, \bullet) + (\bar{d}_0, c_1, \bullet)(\bar{d}_1, \bar{d}_0, c_3, c_2, \bullet), \end{aligned} \tag{31}$$

$$0 = (\bar{d}_0, c_3, c_2, c_1, \bullet)(\bullet) - (\bar{d}_0, c_3, \bullet)(c_2, c_1, \bullet) + (\bar{d}_0, c_2, \bullet)(c_3, c_1, \bullet) - (\bar{d}_0, c_1, \bullet)(c_3, c_2, \bullet), \tag{32}$$

$$\begin{aligned} 0 = (\bar{d}_2, c_3, c_2, c_1, \bar{d}_0, \bar{d}_1, \bullet)(\bar{d}_0, \bar{d}_1, \bullet) - (\bar{d}_2, c_3, \bar{d}_0, \bar{d}_1, \bullet)(c_2, c_1, \bar{d}_0, \bar{d}_1, \bullet) \\ + (\bar{d}_2, c_2, \bar{d}_0, \bar{d}_1, \bullet)(c_3, c_1, \bar{d}_0, \bar{d}_1, \bullet) - (\bar{d}_2, c_1, \bar{d}_0, \bar{d}_1, \bullet)(c_3, c_2, \bar{d}_0, \bar{d}_1, \bullet). \end{aligned} \tag{33}$$

Thus, we have finished the proof.

As an example of pfaffian element (i, j), we can take

$$(i, j) = c_{ij} + \sum_{s=0}^{\infty}(-1)^s\phi_i(s)\bar{\phi}_j(-s - 1) - \sum_{s=0}^{\infty}(-1)^s\phi_j(s)\bar{\phi}_i(-s - 1). \tag{34}$$

The N-soliton solution for equations (17)–(19) is given by τ, σ and $\bar{\sigma}$ in equation (20) with

$$(i,j) = c_{ij} + \frac{1}{p_i+\bar{p}_j}\alpha_i\prod_\nu(1-p_ia_\nu)^{-k_\nu/a_\nu}\bar{\alpha}_j\prod_\mu(1+\bar{p}_ja_\mu)^{k_\mu/a_\mu}$$
$$\qquad - \frac{1}{p_j+\bar{p}_i}\alpha_j\prod_\nu(1-p_ja_\nu)^{-k_\nu/a_\nu}\bar{\alpha}_i\prod_\mu(1+\bar{p}_ia_\mu)^{k_\mu/a_\mu}, \tag{35}$$

$$\phi_i(k_1,k_2,k_3;s) = p_i^s\alpha_i\prod_\nu(1-p_ia_\nu)^{-k_\nu/a_\nu}, \tag{36}$$

$$\bar{\phi}_i(k_1,k_2,k_3;s) = \bar{p}_i^s\bar{\alpha}_i\prod_\nu(1+\bar{p}_ia_\nu)^{k_\nu/a_\nu}. \tag{37}$$

5. Conclusion

From [1, 2, 10], we know that the KP equation has solutions expressed in a Grammian form and accordingly the coupled KP equation possesses the solution expressed in the Gram-type pfaffian. Following the KP equation and the coupled KP equation, we have obtained the Grammian determinant solutions and the Gram-type pfaffian solutions for three pfaffianized differential-difference systems in [11]. As for the fully discrete case, the Grammian determinant solution has been proposed in [12].

In this paper, we first proved the determinant solution to the dKP equation by the algebraic identity of a pfaffian instead of Laplace expansion for determinants. Once we get the pfaffian expression of the solution and the proof of using the pfaffian identities, we can generalize the dKP equation to the coupled dKP by applying the pfaffianization, that is, replacing the pfaffian element for the solution of dKP by the generic one. Then we naturally get the Gram-type pfaffian solution to the pfaffianized dKP system referring to the continuous cases and the semi-discrete ones. It is obvious that the dKP equation and its solution is a special case of the coupled dKP and its pfaffian solution. By taking

$$\phi_i = \bar{\phi}_{2N+1-i} = 0, \qquad i = N+1,\ldots,2N, \tag{38}$$
$$(i,j) = 0, \qquad 1 < i < j < N \qquad \text{or} \qquad N+1 < i < j < 2N, \tag{39}$$

the pfaffian τ in equation (20) reduces to the pfaffian τ in equation (10) and the coupled dKP equation (17) goes down to the dKP equation (3). All equations in section 2 can be derived from the corresponding ones in section 3 just by re-numbering $\bar{\phi}_{2N+1-i} \to \bar{\phi}_i$ for $1 < i < N$ and rewriting $c_\nu \to d_\nu^*$.

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References

[1] Hirota R and Ohta Y 1991 *J. Phys. Soc. Japan* **60** 798–809
[2] Hirota R 1992 *Direct Method in Soliton Theory (in Japanese)* (Tokyo: Iwanami Shoten)
[3] Gilson C R and Nimmo J J C 2001 *Theor. Math. Phys.* **128** 870–82
[4] Ohta Y, Nimmo J J C and Gilson C R 2001 *Glasg. Math. J.* A **43** 99–108
[5] Gilson C R, Nimmo J J C and Tsujimoto S 2001 *J. Phys. A: Math. Gen.* **34** 10569–75
[6] Sato M 1981 *RIMS Kokyuroku* **439** 30–46
[7] Ohta Y, Satsuma J, Takahashi D and Tokihiro T 1988 *Progr. Theor. Phys. Suppl.* **94** 210–41
[8] Freeman N C and Nimmo J J C 1983 *Phys. Lett.* A **95** 1–3
[9] Nimmo J J C and Freeman N C 1983 *Phys. Lett.* A **95** 4–6
[10] Nakamura A 1989 *J. Phys. Soc. Japan* **58** 412–22
[11] Li C X, Hu X B and Zhao J X 2004 *Inverse Problems* **20** 1293–306
[12] Ohta Y, Hirota R, Tsujimoto S and Imai T 1993 *J. Phys. Soc. Japan* **62** 1872–86
[13] Muir T 1882 *A Treatise on the Theory of Determinants* (London: Macmillan)